Lecture 11 on Oct. 21 2013

We already introduced the functions e^z, $\log z$ and z^α. Now we consider the mappings given by these functions and their compositions.

Exponential mapping First of all, we consider e^z. Given line $x = x_0$, we know that the points on this line can be parametrized by $x_0 + iy$, where y is a variable. Therefore under the mapping of e^z, these points are mapped to $e^{x_0}e^{iy}$. Clearly for any y, $e^{x_0}e^{iy}$ lies on the circle C whose center is 0 and radius is e^{x_0}. While y varies from $-\infty$ to ∞, the imaging points circulate along the circle C infinitely many times. Of course, e^z is not 1-1 when restricted on the whole line $x = x_0$. But when we restrict e^z to a segment on $x = x_0$ whose length is smaller than 2π, e^z is a 1-1 mapping. Now we consider the line $y = y_0$. all points on this line are parametrized by $x + iy_0$. Therefore by e^z, these points are mapped to $e^x e^{iy_0}$. Clearly all imaging points share same argument, y_0. While x varies from $-\infty$ to ∞, e^x runs from 0 to ∞. Therefore e^z maps $y = y_0$ to a ray with argument y_0. Notice that the origin is not on the image. Moreover this mapping is 1-1 in that e^x is a one-one function. Now we give two lines $y = y_1$ and $y = y_2$. If $|y_2 - y_1| < 2\pi$, then e^z is 1-1 on the strip L whose boundaries are given by $y = y_1$, $y = y_2$. Moreover e^z send $y = y_1$ to the ray with argument y_1. It also sends $y = y_2$ to the ray with argument $y = y_2$. Furthermore, it sends all points on the strip L to points with argument between y_1 and y_2.

log mapping Since $\log z$ is the inverse function of e^z, we can transform regions between two rays to a strip. Now we use one example to show applications of translation, dilation and $\log z$.

Example 1: transform $A = \{\pi/4 < \theta < 3\pi/4\}$ onto the strip $L = \{1 < y < 2\}$.

Step 1. cut the negative x-axis from the complex plane and let arguments for the remaining points lie on the interval $[-\pi, \pi)$. Using this branch, we define a log function $\log z = \log|z| + i\arg(z)$, where $\arg(z)$ lies in $[-\pi, \pi)$;

Step 2. Using the log function in Step 1, we can map A to the strip $L_1 = \{\pi/4 < y < 3\pi/4\}$. Defining $w_1 = 2z/\pi$, L_1 is mapped to $L_2 = \{1/2 < y < 3/2\}$;

Step 3. Translating L_2 by 1/2 along the positive direction of the y-axis, we are done. So the mapping realizing A to L is $(2\log z)/\pi + i/2$.

power functions Choosing a branch for $\log z$, we can write $\log z = \log|z| + i\arg(z)$. by definition of power functions, we have

$$z^\alpha = e^{\alpha(\log|z| + i\arg(z))} = |z|^\alpha e^{i\alpha\arg(z)},$$

where we assume α is a real number. Therefore we have $|z^\alpha| = |z|^\alpha$, $\arg(z^\alpha) = \alpha\arg(z)$. The main application of power functions is to change argument for points on a ray.

Example 2: Letting A be the region on $\mathbb{C}$ without negative x-axis and B be the right part of the imaginary line, then $z^{1/2}$ sends A to B. Here we choose the same branch for $\log z$ as Step 1 of Example 1. Clearly by this branch, $\arg(z)$ lies in $[-\pi, \pi)$, where z is on A. by definition of power functions, $z^{1/2}$ has argument $\arg(z)/2$. Therefore $\arg(z^{1/2})$ lies on $[-\pi/2, \pi/2)$, for all z in A.

Mapping for $\mathbb{C} \setminus [-1, 1]$ We now try to map $\mathbb{C} \setminus [-1, 1]$ to the interior of the unit disk.

Step 1: Letting $w_1 = (z+1)/(z-1)$, we can map $[-1, 1]$ to the negative part of the x-axis;

Step 2: By the power function in Example 2, we have $w_2 = w_1^{1/2}$, which maps the complement set of the negative part of the x-axis to the right part of the imaginary line;

Step 3: Using $w_3 = (w_2 - 1)/(w_2 + 1)$, we can map the right part of the imaginary line to the interior of the unit disk. Compose the above three steps, the mapping we need is

$$w = \frac{\left(\frac{z+1}{z-1}\right)^{1/2} - 1}{\left(\frac{z+1}{z-1}\right)^{1/2} + 1}. \tag{0.1}$$

The inverse mapping of (0.1) Sovling z in (0.1) by w, one can easiliy show that

$$z = \frac{1}{2}\left(w + \frac{1}{w}\right). \tag{0.2}$$

Now we are going to study this mapping in details. Assuming $w = \rho e^{i\theta}$ and $z = x + iy$, it holds

$$x = \frac{1}{2}\left(\rho + \rho^{-1}\right)\cos\theta, \qquad y = \frac{1}{2}\left(\rho - \rho^{-1}\right)\sin\theta. \tag{0.3}$$

Fixing $\rho_0 < 1$ and letting θ be a variable running from $[0, 2\pi)$, then w variable vary along some circle C with radius ρ_0 once. Here C is centered at the original point. Moreover by (0.3), one can eleminate the variable θ and show that the image of C satisfies the equation

$$\frac{x^2}{\left[\frac{1}{2}\left(\rho_0 + \rho_0^{-1}\right)\right]^2} + \frac{y^2}{\left[\frac{1}{2}\left(\rho_0 - \rho_0^{-1}\right)\right]^2} = 1.$$

Clearly it is an ellipse with foci locating at -1 and 1. Now we begin to decrease ρ from ρ_0 to 0. By this means, we want to find the image of the disk enclosed by C under the mapping (0.2). Noticing that when $\rho < 1$, the function $\rho + \rho^{-1}$ is strictly decreasing. While $\rho \downarrow 0$, we have $\rho + \rho^{-1} \uparrow \infty$. Therefore the major axis of the ellipse is expanding while ρ is decreasing. Samely $\rho^{-1} - \rho$ is also decreasing with respect to ρ. This shows that the minor axis is also expanding while ρ decrease to 0. So we know that while we decrease ρ from ρ_0 to 0, it helps us sweep out the whole exterior part of the ellipse. up to now, we show that (0.2) maps the disk enclosed by C to $\{$the exterior part of the ellipse$\} \cup \{\infty\}$.

Now we fix θ_0 and letting ρ be a variable. Clearly w variable now vary along a ray L with argument θ_0. Still using (0.3) to eleminate variable ρ, we know that the image of the ray L under the mapping (0.2) should satisfy

$$\frac{x^2}{\cos^2\theta_0} - \frac{y^2}{\sin^2\theta_0} = 1. \tag{0.4}$$

It is a hyperbola with two branches. Moreover the associated foci are also located at -1 and 1. Some remarks have to be put here. If we assume θ_0 lies in $(0, \pi/2)$, then by (0.3) x variable is always positive. Therefore the ray L is mapped to the right branch of the hyperbola. Secondly since $\rho - \rho^{-1}$ is an increasing function, y coordinate for imaging point is increasing while we increase ρ from $-\infty$ to $+\infty$. Therefore the mapping (0.2) is one-one mapping from the ray L to the right branch of (0.4). Now we increase θ variable from θ_0 to $\pi - \theta_0$. On $(\theta_0, \pi/2)$, cos function is decreasing. Therefore the imaging hyperbola is moving left-ward while θ is increasing. When $\theta = \pi/2$, the image is the whole imaginary line. On the interval $(\pi/2, \pi - \theta_0)$, cos function is still decreasing but negative. Now the corresponding ray with argument θ is mapped to the left branch of the corresponding hyperbola. Therefore the above arguments show that (0.2) maps the region $\{\theta_0 < \theta < \pi - \theta_0\}$ to the region enclosed by the two branches of the hyperbola (0.4). the mapping is one-one and onto.